This month’s column begins the process of narrowing down the<a href="http://underthehood.blogwyrm.com/?p=1231"> general equations of fluid mechanics</a>, developed over the last many columns, into a set of specific equations. In particular, to make progress in describing specific situations, we must be able to relate the stress tensor to the state of the fluid. This relation is analogous to the<a href="http://underthehood.blogwyrm.com/?p=1161"> generalized Hooke’s law used in the study of elastic deformations</a>.

While<a href="http://underthehood.blogwyrm.com/?p=1240"> there are an amazingly large number of types of fluids out there in the world</a>, this discussion and those that follow will be focused only on the simplest of these: Newtonian fluids for which there is a linear relationship between the applied shearing stress and the velocity gradients that result. To further simplify matters, we shall also look only at the form of the Cauchy momentum equation for a Newtonian fluid, which is known by the famous name of the Navier-Stokes equation. The treatment here parallels the one found in <i>An Introduction to Fluid Dynamics</i>, by G.K. Batchelor.

There are interesting differences between a moving fluid and one at rest. A fluid at rest necessarily exerts a normal stress independent of direction; no shear stresses are supported and the stress must to isotropic, leading to a very simple expression for the stress tensor

\[ T\_{ij} = -p \delta\_{ij} \; . \]

The parameter $$p$$ is the static fluid pressure, which can only be a function of position (e.g. the change in pressure with depth in a swimming pool).

Batchelor notes that there should be no expectation that this simple relation holds for a moving fluid, since, once in motion, the fluid can support shearing stress and will, in general, depart from isotropy. Nonetheless, it helps to have a scalar quantity, similar to static pressure, that characterizes the “squeezing” of the fluid. This pressure is defined by averaging the normal component of stress over the surface of a small sphere centered on the position $$\vec x$$

\[ P = -\frac{1}{3} T\_{ii} = - \frac{1}{3} T\_{ij} \delta\_{ij} = -\frac{1}{4 \pi} T\_{ij} \int n\_i n\_j d \Omega(\vec n) \; , \]

where $$\vec n$$ is the normal to the surface of the sphere. Batchelor emphasizes that $$P$$ is a mechanical definition with only a loose connection with thermodynamic notion of pressure, the latter requiring some notion of equilibrium, which is absent when the fluid is in motion. Nonetheless, $$P$$ is a useful quantity because it is experimentally accessible from measurements of force and momentum (or, more accurately, from typical mechanical measurements that allow the inference of force and momentum).

Since $$P$$ is associated with the isotropic portion of the stress, it is mathematically convenient to decompose the stress tensor as

\[ T\_{ij} = -P \delta\_{ij} + d\_{ij} \; \]

where $$d\_{ij}$$ is a traceless tensor, known as the deviatoric stress, whose non-zero nature is due solely to the fluid motion. It represents the internal transport of momentum, which can manifest itself as friction.

It is at this point, where the assumption that the fluid is Newtonian is brought to bear by expressing the deviatoric stress as being linearly proportional to the velocity gradient by

\[ d\_{ij} = A\_{ijk \ell} \frac{\partial V\_k}{\partial x\_\ell} \; \]

Where the tensor $$A\_{ijk \ell}$$ depends on the local state of the fluid but not directly on the velocity distribution. Since the deviatoric stress is symmetric in its indices we can conclude that $${\mathbf A}$$ is symmetric in its first two indices.

Next, we can decompose the velocity gradient as

\[ \frac{\partial V\_k}{\partial x\_{\ell}} = e\_{k \ell} - \frac{1}{2} \epsilon\_{k \ell m} \omega\_m \; , \]

where $$e\_{k \ell} = 1/2 (\partial\_k V\_\ell + \partial\_\ell V\_k)$$ is a symmetric tensor and $$\omega\_m$$ is the vorticity defined, in terms of the anti-symmetric portion of the velocity gradient, by

\[ \omega\_m = \epsilon\_{m n p} \frac{1}{2} \left(\partial\_n V\_p - \partial\_p V\_n \right) \; .\]

The penultimate simplifying assumption in this long cavalcade of reductions, is to confine the discussion to gases and simple liquids, in which the ‘shape’ of the molecules is nearly spherical. This assumption allows us to conclude that the deviatoric stress is independent of the orientation of the fluid element, thus imposing rotational symmetry on $$A\_{i j k \ell}$$. This assumption should be viewed in contrast to a fluid whose molecular constituents are long chains and that, as a result, exhibits a preferred (i.e. easy) axis.

Imposition of rotational symmetry on $${\mathbf A}$$ means that it can only have the following general form

\[ A\_{i j k \ell} = \mu \delta\_{ik} \delta\_{j \ell} + \mu' \delta\_{i\ell} \delta\_{jk} + \mu'' \delta\_{ij}\delta\_{kl} \; .\]

The ‘mu’ parameters describe the viscosity that arises from various shearing motions. Because of the symmetries imposed on $${\mathbf d}$$ and upon $${\mathbf A}$$, they are not all independent. Since $${\mathbf A}$$ is symmetric in its first two indices $$ \mu = \mu'$$ and

\[ A\_{i j k \ell} = \mu ( \delta\_{ik} \delta\_{j \ell} + \delta\_{i \ell} \delta\_{jk} )+ \mu'' \delta\_{ij} \delta\_{k\ell} = \mu (\delta\_{i \ell} \delta\_{jk} + \delta\_{ik} \delta\_{j\ell} ) + \mu'' \delta\_{ij}\delta\_{ \ell k} = A\_{ij \ell k} \; ,\]

Clearly showing that $${\mathbf A}$$ is now symmetric in its third and fourth indices as well. Note that this symmetry was made manifest by swapping the first and second terms and using the symmetry of the Kronecker delta in the term.

Armed with this knowledge of $${\mathbf A}$$, we return to the relation between the deviatoric stress and the velocity gradient to find

\[ d\_{ij} = A\_{ijk \ell} \frac{\partial u\_k}{\partial x\_{\ell}} = A\_{ijk \ell} e\_{k \ell} = \left[ \mu(\delta\_{ik} \delta\_{j\ell} + \delta\_{i\ell} \delta\_{jk}) + \mu'' \delta\_{ij} \delta\_{k\ell} \right] e\_{k\ell} = 2 \mu e\_{ij} + \mu'' Tr({\mathbf e} \delta\_{ij}) \; . \]

Note that the second equality follows from the symmetry of $${\mathbf A}$$ in its last two indices, the third equality by substituting in the specific form of $${\mathbf A}$$ and the final one by multiplying the contractions out and defining $$e\_{kk} \equiv Tr({\mathbf e})$$. The final step is to note that, since $${\mathbf d}$$ is traceless

\[ Tr({\mathbf d}) = 2 \mu Tr({\mathbf e}) + 3 \mu'' Tr({\mathbf e}) = 0 \; \]

implying that $$\mu'' = - 2/3 \mu$$.

After all this work, we now have a form of the deviatoric stress $${\mathbf d}$$ in terms of the symmetric part $${\mathbf e}$$ of the velocity gradient given by

\[ d\_{ij} = 2 \mu \left( e\_{ij} - \frac{1}{3} Tr({\mathbf e}) \delta\_{ij} \right) \; . \]

This expression was the result of many hands in the mid-1800s, but often bears the name of Saint-Venant who derived essentially as was done above in 1843.

Suppose that the only non-zero velocity gradient component is $$\partial u\_x/\partial y$$. All components of deviatoric stress is zero except for the tangential stresses $$d\_{xy} = d\_{yx} = \mu \partial u\_x/\partial y$$, which justifies identifying $$\mu$$ with the elementary definition of viscosity.

The Cauchy momentum equation (in conservative form) is now

\[ \frac{\partial \rho \vec V}{\partial t} + \nabla \cdot (\rho \vec V \vec V) = \nabla \cdot \left[ -p {\mathbf 1} + 2 \mu \left( e\_{ij} - \frac{1}{3} Tr({\mathbf e}) \delta\_{ij} \right) \right] + \vec f\_b \; . \]

Finally, assume that the viscosity is a constant (that is we will initially confine ourselves to those cases where this is a very good approximation), to get the Navier-Stokes equations

\[ \frac{\partial \rho \vec V}{\partial t} + \nabla \cdot (\rho \vec V \vec V) = - \nabla p + \mu \nabla^2 \vec V + \frac{1}{3} \mu \nabla (\nabla \cdot \vec V) + \rho \vec g \; , \]

where we used $$Tr({\mathbf e}) = \nabla \cdot \vec V$$ and we specialized to gravity as the only body force: $$\vec f\_b = \rho \vec g$$. These cosmetic changes are design to facilitate comparison with the usual way the Navier-Stokes equations are written.

Next column will begin looking at information that can be interrogated from these equations and the solutions in simple situations.